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Finite-Temperature Gravitational Čerenkov Radiation

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The graviton action in vacuum is generalized for a medium with a constant gravitational index of refraction. From this generalized action the power spectrum of the Čerenkov radiation of gravitons is derived in the framework of source theory at zero and nonzero temperature.

1. INTRODUCTION

Čerenkov radiation is produced in electrodynamics by a fast-moving charged particle in a medium when its speed is faster than the speed of light in this medium. This radiation was first observed experimentally by Čerenkov (1934) and theoretically interpreted by Frank and Tamm (1937) in the framework of classical electrodynamics. The source-theoretic description of this effect was given by Schwinger *et al.* (1976) for zero temperature, and the classical spectral formula was generalized in source theory at finite temperature by Pardy (1989).

Gravitational Čerenkov radiation as the analog of the electromagnetic effect is obviously conditioned by the gravitational index of refraction. There are a number of discussions concerning of the propagation of gravitational waves in bulk matter with a gravitational index of refraction. Szekeres (1971) found the index of refraction of gravitational waves propagating through matter composed of particles in which the incident wave induces quadrupole moments. Polnarev (1972) and Chesters (1973) discussed the interaction of gravitational waves in a hot gas and Peters (1974) calculated the index of refraction of a cold gas of free particles.

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In classical electrodynamics, the existence of Čerenkov radiation is a natural consequence of the existence of an index of refraction of the medium. In the analogous gravitational situation the gravitational Čerenkov radiation is a natural consequence of the existence of a gravitational index of refraction.

Here we do not consider the microscopic mechanism generating the gravitational index of refraction, however. We define the index of refraction by the metric $g_{\mu\nu}$ involved in the equation for the Green function in the background gravitational field with metric $g_{\mu\nu}$:

$$\partial_\mu(\sqrt{-g}\partial^\mu)D_{+g}(x) = -\frac{1}{\sqrt{-g}}\,\delta(x) \tag{1}$$

where g is the determinant of the $g_{\mu\nu}$. Now, if we define the background metric by the equations

$$g_{k0} = 0, \qquad g_{kl} = \delta_{kl}, \qquad g_{00} = -n^2 \tag{2}$$

then the left side of equation (1) is just the left side of the wave equation with the index of refraction n, and obviously the Green function defined by (1) is the Green function for propagation of massless particles in the background medium with velocity $c' = c/n$ and not c.

Under such conditions, we derive in this article the power spectrum of gravitons in the framework of the Schwinger source theory (Schwinger *et al.*, 1976; Schwinger, 1970) at zero temperature, and using the finite-temperature graviton propagator, we generalize the result for the nonzero-temperature situation.

First, we generalize the graviton action to the situation with the general metric $g_{\mu\nu}$ and then we specify the metric by relations (2). The derivation of the power spectrum is analogous to the electromagnetic case. The result is the gravitational analog of the Frank–Tamm formula for electromagnetic Čerenkov radiation. The finite-temperature gravitational Čerenkov radiation is derived here by the finite-temperature procedure (Pardy, 1989).

2. SOURCE-THEORY FORMULATION OF THE PROBLEM IN RIEMANN SPACE-TIME

Source theory (Schwinger *et al.*, 1976; Schwinger, 1970) is a theoretical construction which uses quantum mechanical particle language. Initially it was constructed for a description of particle physics situations in high-energy physics experiments. However, it was found that the original formulation simplifies calculations in electrodynamics and gravity, where the interactions are mediated by the photon or the graviton, respectively. The special values of mass and spin of the photon or the graviton combined with the general laws of quantum mechanics and special relativity are so restrictive that the essential frameworks of these fundamental theories are so analogous that it is possible to speak of the methodological unification of electromagnetism and gravity (Schwinger, 1976). This means that the analogy can be expected also in the specific situation of production of gravitons by the motion of particles in a medium of gravitational index of refraction n.

The basic formula of the source theory is the vacuum-to-vacuum amplitude (Schwinger, 1970):

$$\langle 0_+ | 0_- \rangle = e^{(i/\hbar)W(S)} \tag{3}$$

where the minus and plus tags on the vacuum symbol are causal labels, referring to any time before and after the space-time region where the sources are manipulated. The exponential form is introduced with regard to the existence of physically independent experimental arrangements and has a simple consequence that the associated probability amplitudes multiply and the corresponding W expressions add (Schwinger *et al.*, 1976; Schwinger, 1970).

In the flat space-time the field of gravitons is described by the amplitude (3) with the action

$$\begin{aligned} W(T) &= \frac{4\pi G}{c^4} \int (dx)(dx')[T^{\mu\nu}(x)D_+(x - x')T_{\mu\nu}(x') \\ &\quad - \frac{1}{2} T(x)D_+(x - x')T(x')] \end{aligned} \tag{4}$$

where the dimensionality of $W(T)$ is the same as the dimensionality of the Planck constant $\hbar$. Here $T_{\mu\nu}$ is the tensor of momentum and energy, and for a particle moving along the trajectory $\mathbf{x} = \mathbf{x}(t)$ it is defined by the equation

$$T^{\mu\nu}(x) = c^2 \frac{p^\mu p^\nu}{E} \delta(\mathbf{x} - \mathbf{x}(t)) \tag{5}$$

where p^μ is the relativistic four-momentum of a particle with a rest mass m and

$$p^\mu = (E/c, \mathbf{p}) \tag{6}$$

$$p^\mu p_\mu = -m^2c^2 \tag{7}$$

and the relativistic energy is defined by

$$E = \frac{mc^2}{(1 - \mathbf{v}^2/c^2)^{1/2}} \tag{8}$$

where $\mathbf{v}$ is the three-velocity of the moving particle.

The symbol $T(x)$ in formula (4) is defined as $T = g_{\mu\nu}T^{\mu\nu}$ and the symbol $D_+(x - x')$ is the graviton propagator; its explicit form will be determined later.

In the case of nonflat space-time with the general metric $g_{\mu\nu}$ there exists a system of rules to transcribe the action $W(T)$. It follows from general relativity theory (Weinberg, 1972) that all equations and formulas are influenced by gravity in the presence of the gravitational field expressed by the metric tensor $g_{\mu\nu}$. The general method for including the effect of gravity on mechanics and electrodynamics consists first in formulating the equations of motion from the viewpoint of the special theory of relativity and then formulating them in a general covariant way which is equivalent to the situation with the gravitational field on the condition that the system is sufficiently small in comparison with the scale of the fields. According to Weinberg, the rules generating the general covariance are as follows:

$$(dx) \to \sqrt{-g}\,(dx) \tag{9}$$

$$T_{\mu\nu} \to \frac{1}{\sqrt{-g}}\,T_{\mu\nu} \tag{10}$$

$$T_{\mu\nu} \to g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta} \tag{11}$$

$$D_+ \to D_{+g}(x, x') \tag{12}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$. The function $D_{+g}(x, x')$ is the graviton propagator in the gravitational field and in our case it is the graviton propagator in the metric corresponding to the gravitational index of refraction n.

In this way we get the action $W(T)$ embedded into the space-time with metric $g_{\mu\nu}$:

$$W(T) = \frac{4\pi G}{c^4}\int (dx)(dx')\Big[T^{\mu\nu}(x)g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}(x')D_{+g}(x, x') - \frac{1}{2}\,g_{\mu\nu}T^{\mu\nu}(x)D_{+g}(x, x')g_{\alpha\beta}T^{\alpha\beta}(x')\Big] \tag{13}$$

The formula (13) describes the interaction of a particle with zero mass and spin 2 and spirality ± 2 (graviton) with the metric field of external gravity. The derivation of the general covariant action is in agreement with the discussion in Yilmaz (1975) concerning gravity and source theory.

3. THE POWER SPECTRAL FORMULA

The probability of the persistence of the vacuum is given by (Schwinger *et al.*, 1976)

$$|\langle 0_+|0_-\rangle|^2 = \exp\left(-\frac{2}{\hbar}\,\mathrm{Im}\,W\right) \stackrel{d}{=} \exp\left[-\int dt\,d\omega\,\frac{c}{\hbar\omega}\,P(\omega, t)\right] \tag{14}$$

where we have introduced the so-called power spectral function (Schwinger *et al.*, 1976) $P(\omega, t)$. In order to extract this spectral function from Im W, it is necessary to know the explicit form of the graviton propagator $D_{+g}(x - x')$. The physical content of this propagator is analogous to the photon propagator. It involves the property of spreading of gravitons with velocity c/n. Its explicit form is the same as that of the photon propagator. With regard to Schwinger *et al.* (1976) and (1) with metric (2), we can therefore write for our problem

$$\begin{aligned} D_{+g}(x - x') &= \frac{1}{n^2}\int_0^\infty \frac{(dk)}{(2\pi)^4}\,\frac{e^{ik(x-x')}}{|\mathbf{k}|^2 - n^2(k^0)^2 - i\epsilon} \\ &= \frac{i}{4\pi^2 cn^2}\int_0^\infty d\omega\,\frac{\sin(n\omega/c)|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|}\,e^{-i\omega|t-t'|} \end{aligned} \tag{15}$$

Now, using formulas (13)–(15), we get the power spectral formula in the form

$$\begin{aligned} P(\omega, t) = \frac{4\pi G}{c^4 n^2}\int (d\mathbf{x})(d\mathbf{x}')dt'\,\frac{\sin(n\omega/c)|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|}\cos\omega(t - t') \\ \times\left[T^{\mu\nu}(\mathbf{x}, t)g_{\mu\alpha}g_{\nu\beta}T^{\alpha\beta}(\mathbf{x}', t') - \frac{1}{2}\,g_{\mu\nu}T^{\mu\nu}(\mathbf{x}, t)g_{\alpha\beta}T^{\alpha\beta}(\mathbf{x}', t')\right] \end{aligned} \tag{16}$$

Čerenkov radiation in electrodynamics is produced in the linear case by a uniformly moving charge with constant velocity $\mathbf{v} = (v, 0, 0)$. In the gravitational situation gravitational Čerenkov radiation is generated by the energy-momentum tensor of a uniformly linearly moving particle with rest mass m and constant velocity $\mathbf{v}$. If we insert the tensor of the energy-momentum of the particle moving along the trajectory $\mathbf{x} = \mathbf{v}t$ into (16), then using the metric tensor (2), $\tau = t - t'$, and $\beta = v/c$, we get instead of the formula (16) the relation

$$P(\omega, t) = \frac{G\omega}{\pi v n^2}\,\frac{m^2}{1 - \beta^2}\,\beta^4\left(1 + \frac{n^2}{\beta^2}\right)^2 \int_{-\infty}^{\infty} d\tau\,\frac{\sin n\omega\beta\tau}{\tau}\cos\omega\tau \tag{17}$$

The formula (17) contains the known integral

$$\int_{-\infty}^{\infty} d\tau\,\frac{\sin n\omega\beta\tau}{\tau}\cos\omega\tau = \begin{cases} \pi, & n\beta > 1 \\ 0, & n\beta < 1 \end{cases} \tag{18}$$

Using the integral (18), we finally get the power spectral formula of the produced gravitons:

$$P(\omega, t) = \frac{G\omega}{vn^2}\frac{m^2}{1-\beta^2}\beta^4\left(1+\frac{n^2}{\beta^2}\right)^2; \qquad n\beta > 1 \tag{19}$$

and $P(\omega, t) = 0$ for $n\beta < 1$.

The power spectral formula (19) is the gravitational analog of the Frank–Tamm formula for Čerenkov radiation in electrodynamics.

The dimensionality of $P(\omega, t)$ is ergs because $[G] = \text{cm}^3\ \text{g}^{-1}\ \text{s}^{-2}$, $[\omega] = \text{s}^{-1}$, $[m^2] = \text{g}^2$, and $[v^{-1}] = \text{cm}^{-1}\ \text{s}$. This is in agreement with the definition of the power spectral formula involved in the energy loss equation of the produced radiation:

$$-\frac{dE}{dt} = \int d\omega\, P(\omega, t) \tag{20}$$

4. FINITE-TEMPERATURE CONTRIBUTION

Finite-temperature quantum field theory (QFT) was developed two decades ago and is being intensively studied. The first formulation of finite-temperature QFT was presented by Dolan and Jackiw (1974), Weinberg (1974), and Bernard (1974) and its first application concerned the effective potential in Higgs theories.

Quantum chromodynamics (QCD) was also studied at finite temperature and densities using the temperature Green functions (Kalashnikov, 1984). The systematic examination of finite-temperature effects in quantum electrodynamics (QED) at one-loop order was elaborated by Donoghue *et al.* (1985) and Johansson *et al.* (1986). The finite-temperature Čerenkov electrodynamic power spectral formula in source theory was also derived (Pardy, 1989).

Here we use the Pardy procedure in order to generalize the formula (19) to the finite-temperature regime. It consists in the real-time formulation in the following transformation in the graviton propagator (15):

$$\frac{1}{|\mathbf{k}|^2 - n^2(k^0)^2 - i\epsilon} \rightarrow \frac{1}{|\mathbf{k}|^2 - n^2(k^0)^2 - i\epsilon} + \frac{2\pi i}{e^{|E|/k_BT} - 1}\,\delta(|\mathbf{k}|^2 - n^2(k^0)^2) \tag{21}$$

where $E = \hbar\omega$ is the energy of the graviton, k_B is the Boltzmann constant, and T is the temperature of the graviton gas in the gravitational medium with the index of refraction n. The considered situation is the analog of the electrodynamic one.

The transformation (21) enables us immediately to separate the finite-temperature part of the Green function. After inserting (21) into (15) we get, using some obvious mathematical operations, the temperature part of the D_{+g}-function in the form

$$D_{+gT}(x - x') = \frac{i}{2\pi^2 c n^2} \int_0^\infty d\omega \frac{\sin(n\omega/c)|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|} \frac{\cos \omega(t - t')}{\exp(\hbar\omega/k_B T) - 1} \tag{22}$$

It is obvious that D_{+gT} is pure imaginary. Using definition (14), we get for the finite-temperature part of the spectral function the formula

$$P_T(\omega, t) = \frac{2}{\exp(\hbar\omega/k_B T) - 1} \times \frac{4\pi G}{c^4 n^2} \int (d\mathbf{x})(d\mathbf{x}')dt' \frac{\sin(n\omega/c)|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|} \cos \omega(t - t') \times \left[T^{\mu\nu}(\mathbf{x}, t) g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}(\mathbf{x}', t') - \frac{1}{2} g_{\mu\nu} T^{\mu\nu}(\mathbf{x}, t) g_{\alpha\beta} T^{\alpha\beta}(\mathbf{x}', t') \right] \tag{23}$$

The last formula differs from the zero-temperature formula only by the multiplicative factor $2/[\exp(\hbar\omega/k_B T) - 1]$. The total spectral formula is given obviously by the relation

$$P_{\text{total}} = P_{T=0} + P_T = P_{T=0}\left(1 + \frac{2}{\exp(\hbar\omega/k_B T) - 1}\right) \tag{24}$$

or, after some algebra and using formula (19),

$$P_{\text{total}} = \frac{G\omega}{v n^2} \frac{m^2}{1 - \beta^2} \beta^4 \left(1 + \frac{n^2}{\beta^2}\right)^2 \coth\left(\frac{\hbar\omega}{2k_B T}\right); \qquad n\beta > 1 \tag{25}$$

and $P_{\text{total}} = 0$ for $n\beta < 1$.

The power spectral formula (25) is the finite-temperature generalization of the power spectral formula for the zero-temperature gravitational Čerenkov radiation (19). This formula was never derived in conventional gravity and is original in the Schwinger source theory.

5. DISCUSSION

The power spectral formulas of gravitational Čerenkov radiation at zero temperature (19) and at nonzero temperature (25) are derived here in the framework of the source theory for the first time. These effects are not discussed in classical textbooks on gravity. Nevertheless these effects can be mathematically rigorously defined and described in the framework of source theory embedded in the curved space-time with metric (2).

Formula (13) is valid for general metric space-time and it enables us to determine the power spectral formula of gravitons for general metric space-time if we know the propagator $D_{+g}(x, x')$. This means it generates further problems of the production of gravitons in different metric space-times.

In electrodynamics, the Čerenkov effect usually occurs for velocities comparable with velocities of light. However, if we consider the cold gas (Peters, 1974), then the Čerenkov gravitational effect occurs practically for all velocities. In order to see this surprising result, let us write the gravitational index of refraction derived by Peters (1974) for the cold gas,

$$n = 1 + \frac{2\pi\rho G}{\omega^2} \tag{26}$$

where ρ is the gas density. Then from the condition $n\beta > 1$ we get, using (21), the inequality

$$\omega < \left(\frac{2\pi\beta\rho G}{1 - \beta}\right)^{1/2} \tag{27}$$

which means that the interval of frequencies is limited and because ρ is very small for the cold gas, gravitational Čerenkov radiation occurs only for very low frequencies. On the other hand, the interval of allowed frequencies is greater for sufficiently fast moving bodies.

The amount of gravitons produced by the Čerenkov mechanism depends on the square of the relativistic mass m, and it is obvious that for elementary particles such as electrons, protons, and so on the production of gravitons will be small.

It is obvious that such a small energy cannot be observed by any experimental equipment. On the other hand, a great production of gravitons by the Čerenkov mechanism can occur for cosmological bodies with sufficiently large masses and with energies exceeding the Čerenkov threshold. Of course, whether such an effect occurred during the explosion of supernova SN 1987a or during the big bang or occurs during the collisions of galaxies is an open question because of the nonexistence of the cold gas. On the other hand, Polnarev (1972) has shown that in the ultrarelativistic case in the anisotropic situation there exists the possibility of $n > 1$ or $n < 1$ and the effect is probable.

The investigation of the gravitational Čerenkov effect is analogous to the historical situation when Sommerfeld (1904, 1905) considered the theory of a charge moving with velocity greater than the velocity of light in vacuum. His theory was not accepted in his time because of the priority of the special theory of relativity, where the maximal velocity is the velocity of light. We hope in this article that the sympathy to the existence of the gravitational Čerenkov radiation will be sufficiently strong in order to have some followers.

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